

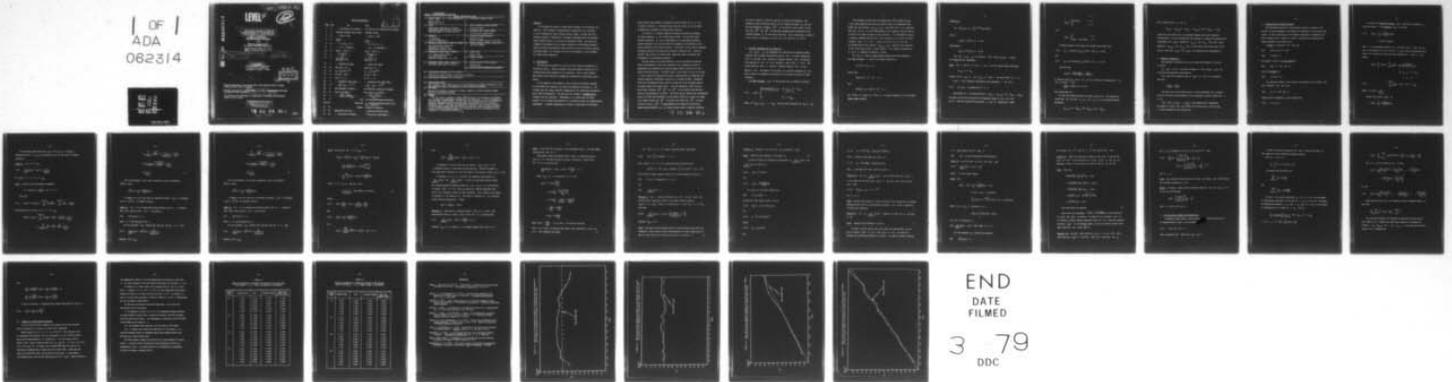
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LARGE SAMPLE ESTIMATES AND UNIFORM CONFIDENCE BOUNDS FOR THE FA--ETC(U)  
AUG 78 J SETHURAMAN, N D SINGPURWALLA N00014-77-C-0263  
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(6) LARGE SAMPLE ESTIMATES AND UNIFORM CONFIDENCE BOUNDS FOR THE FAILURE RATE FUNCTION BASED ON A NAIVE ESTIMATOR.

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List of Corrections

Page	Line	For	Read
3	6	... the failure are studied ...	the failure rate functions are studied.
6	6	Standard extreme value theory..	Standard theory ...
6	13	... given i that	... given i, that
6	14	$\frac{1}{n_r(x)} \dots$	$\frac{1}{r_n(x)} \dots$
8	13	... $0 \leq w(u) \leq M < \infty$ .	... $0 \leq w(u) \leq c < \infty$ .
9	5	... (i.e., satisfies...)	... (i.e., it satisfies ...)
10	12	$ R_n(x) - H_n(x) \leq \dots$	$ R_n(x) - H_n(x)  \leq \dots$
12	2	... $\int w \frac{x-s}{b_n} d[\dots$	... $\int w(\frac{x-s}{b_n}) d[\dots$
12	6	$\frac{1}{2nb_n \bar{F}_n(K+A)}$	$\frac{1}{2nb_b \bar{F}_n(K+A)}$
16	1	... replaced by $\bar{\epsilon}_n(s)$ .	... replaced by $\bar{\epsilon}_n(x)$ .
16	14	$\max \epsilon_n(x)$ , we ...	$\max  \epsilon_n(x) $ , we ...
16	16	$\beta_n = c_n / (\lambda/w)^{1/2}$ ,	$\beta_n = c_n / (\lambda(w))^{1/2}$ ,
17	6	or (W1) - W(4) ...	or (W1) - (W4) ...
17	9	for $0 < x < \infty$ .	for $0 < z < \infty$ .
19	2	... assumption (B3) holds...	... assumption that (B3) holds...
20	3	... $(\frac{z}{n} + \alpha_n) \rightarrow \bar{e} 2^{-z}$ ,	... $(\frac{z}{\beta_n} + \alpha_n) \rightarrow e^{-2e^{-z}}$ ,
20	6	are uniformly estimates...	are uniformly consistent estimates
20	15	This w satisfies...	This window w satisfies...
22	6	... $+\frac{X_{(i)} - X_{(i-1)}}{(n-i+1)U_{(i)}} + \dots$	... $+\frac{X_{(i)} - X_{(i-1)}}{U_{(i)}} + \dots$
27	9		Add: The George Washington University.
27 13, 14			Underline: Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32
27	16		Remove: Program in Logistics,
27	21	generalized end rate...	... generalized failure rate ...
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Abstract.

In this paper we propose a simple naïve estimator of the failure rate function. This estimate is asymptotically unbiased but not consistent. It can be smoothed by using any band limited window. We show that this smoothed estimate is equivalent to estimates obtainable from the modified sample hazard function, as in Rice and Rosenblatt (1976). We obtain the asymptotic distribution of the global deviation of the smoothed estimate from the failure rate function, which can then be used to construct uniform confidence bands. We illustrate the rate of convergence of our estimator by a Monte-Carlo simulation.

1. Introduction.

The failure rate function is one of the most important parameters in reliability theory. Several parametric and non-parametric methods for its estimation have been proposed in the literature. Most of these methods make specific and restrictive assumptions concerning the underlying distribution.

In this paper we first propose a very intuitive (and naïve) estimate of the failure rate function, and study its properties in Section 2. Our estimate does not require specific assumptions on the underlying distribution and is thus non-parametric. We show that our estimate is asymptotically unbiased, but not consistent. Furthermore, it is shown that the naïve estimate of the failure rate function at two distinct points are asymptotically independent. A paradox concerning the estimate, reminiscent of the famous

inter-arrival time paradox illustrated by Feller [(1966) Vol. 2, p. 11], is shown in Section 3. The above facts limit any direct use of the naïve estimator for estimating the failure rate function.

In Section 4, we propose smoothed estimators obtained by averaging the naïve estimator  $r_n$  by a band-limited window. We show that these smoothed estimators  $\bar{r}_n$  can be approximated by an appropriate Gaussian process (Theorem 4.5), and thus obtain the asymptotic distribution of the global deviation on any finite interval. This result can be used to construct confidence bands for the failure rate function (Theorem 4.10). Section 5 contains some results of a Monte-Carlo experiment which illustrates the performance of our smoothed estimators.

The main steps in the proof Theorem 4.5 may be described as follows. Bickel and Rosenblatt (1973) have pioneered a new technique of proof to approximate the normalized and centered sample density function by a stationary Gaussian process. The basic step in their proof is a use of the then available result of Breiman and Brillinger approximating the normalized and centered empirical distribution function by a Brownian bridge. Rosenblatt (1976) has strengthened these results by using the recent stronger results of Komlós, Major and Tusnády (1975). Rice and Rosenblatt (1976) have proposed three estimates  $h_n^{(1)}$ ,  $h_n^{(2)}$ , and  $h_n^{(3)}$  of the failure rate function which are non-parametric in nature. They have directly applied the strengthened results of Bickel and Rosenblatt on the density function to obtain the asymptotic global results for  $h_n^{(1)}$ . They have also shown that  $h_n^{(2)}$  is asymptotically close to  $h_n^{(3)}$ . By approximating the normalized and centered sample hazard functions  $H_n$  by a Weiner process under a monotone transform

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of time, we obtain, in much the same way as Bickel and Rosenblatt, the asymptotic global deviation results for our smoothed estimate  $\bar{r}_n$  and the Rice and Rosenblatt estimator  $h_n^{(3)}$ . In the course of this proof, we also show that  $h_n^{(2)}$  and  $h_n^{(3)}$  are uniformly asymptotically equivalent to our smoothed estimate  $\bar{r}_n$  on each finite interval. Other nonparametric estimates of the failure are studied in Shaked (1978) and Ahmad and Lin (1977).

## 2. The Naïve Estimate and its Properties.

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables with a common distribution function  $F(x)$ . We shall assume that  $F(0) = 0$  and that  $F(x)$  possesses a density function  $f(x)$ . The failure rate function of  $F(x)$  is  $r(x) = f(x)/\bar{F}(x)$  where  $\bar{F}(x) = 1 - F(x)$ . The hazard function  $H(x)$  is  $-\log \bar{F}(x)$  (defined whenever  $\bar{F}(x) > 0$ ) and  $\frac{d}{dx} H(x) = r(x)$ . The purpose of this paper is to provide estimates of  $r(x)$  and to obtain the asymptotic distribution of the global deviation of these estimates.

The naïve estimate  $r_n(x)$  of the failure rate is defined as follows:

$$(2.1) \quad r_n(x) = \begin{cases} \frac{1}{(n-i+1)(X_{(i)} - X_{(i-1)})}, & \text{if } X_{(i-1)} \leq x < X_{(i)} \\ & i = 1, 2, \dots, n \\ 0 & , \text{ if } x \geq X_{(n)}, \end{cases}$$

where  $0 = X_{(0)} \leq X_{(1)} \leq \dots \leq X_{(n)}$  are the order statistics of  $X_1, X_2, \dots, X_n$ .

This estimate has been used by Singpurwalla (1975) wherein he used a time series approach for analyzing failure rates in contaminated data. Before the time interval  $[X_{(i-1)}, X_{(i)}]$  there have been  $(i-1)$  failures; thus only  $(n-i+1)$  are still functioning in this interval, and one item is to fail at the right end point of this interval. It is therefore intuitive to assign a constant failure rate in this interval and to define the naïve estimator  $r_n(x)$  as in (2.1). Another motivation for (2.1) is that the failure rate in the interval  $[X_{(i-1)}, X_{(i)}]$  should be the reciprocal of the 'total time on test' in that interval. The limiting distribution of  $r_n(x)$  is obtained in Theorem 2.1 below.

Let  $G(\alpha; \lambda)$  denote a gamma random variable with scale parameter  $\alpha$  and shape parameter  $\lambda$ ; that is the density function is

$$\alpha^\lambda e^{-x\alpha} x^{\lambda-1} / \Gamma(\lambda), \quad x > 0.$$

Notice that

$$E(G(\alpha; \lambda)) = \lambda/\alpha \quad \text{for } \lambda > 0$$

and

$$E(1/G(\alpha; \lambda)) = \alpha/(\lambda-1) \quad \text{for } \lambda > 1.$$

Let  $G^{-1}(\alpha; \lambda) = 1/G(\alpha; \lambda)$ ;  $G^{-1}(\alpha; \lambda)$  is usually referred to as the inverse gamma random variable.

Theorem 2.1.

$$(i) \quad P(1/r_n(x) \leq a) \rightarrow \int_0^a e^{-yr(x)} y(r(x))^2 dy;$$

that is

$$1/r_n(x) \rightarrow G(r(x), 2) \text{ in law.}$$

Consequently

$$r_n(x) \rightarrow G^{-1}(r(x), 2) \text{ in law.}$$

(ii) Let  $x_1, x_2, \dots, x_k$  be distinct. Then  $\{r_n(x_1), r_n(x_2), \dots, r_n(x_k)\}$  are asymptotically independent.

Proof. Fix  $x$ , and let  $0 < F(x) < 1$ . Let  $i$  be the random suffix satisfying

$$X_{(i-1)} \leq x < X_{(i)}$$

wherein, we set  $X_{(0)} = 0$ , and  $X_{(n+1)} = \infty$ ; thus  $i$  can take values  $1, 2, \dots, n$ .

Since  $i$  has a binomial distribution with parameters  $n$  and  $F(x)$ ,

$$(2.2) \quad \frac{i}{n} \rightarrow F(x), \text{ in probability as } n \rightarrow \infty.$$

Conditional on  $i$ , the distribution of  $(X_{(1)}, \dots, X_{(i-1)})$  and  $(X_{(i)}, \dots, X_{(n)})$  are those of the order statistics of independent sample of size  $(i-1)$  and  $(n-i+1)$  from the truncated distributions  $F_x$  and  $F^x$ , respectively, where

$$F_x(y) = \begin{cases} F(y)/F(x) & y \leq x \\ 1 & y > x \end{cases}$$

and

$$F^x(y) = \begin{cases} 0 & y \leq x \\ (F(y) - F(x))/\bar{F}(x) & y > x. \end{cases}$$

Standard extreme value theory for extreme values shows that

$$(2.3) \quad A_n = (i-1)f(x)(x-X_{(i-1)})/F(x) \rightarrow G(1, 1) \text{ in law}$$

and

$$(2.4) \quad B_n = (n-i+1)f(x)(X_{(i)}-x)/\bar{F}(x) \rightarrow G(1, 1) \text{ in law.}$$

Note now that

$$1/r_n(x) = \frac{n-i+1}{i-1} \frac{F(x)}{f(x)} A_n + \frac{\bar{F}(x)}{f(x)} B_n.$$

It follows from (2.2), (2.3), (2.4) and the conditional independence of  $A_n$  and  $B_n$  given  $i$  that

$$\frac{1}{r_n(x)} \rightarrow G(r(x), 2), \text{ in law.}$$

This proves part (i).

We will now briefly indicate the proof of part (ii). For simplicity, assume that  $k=2$ , and that  $x_1 < x_2$ . Let  $i_1 \leq i_2$  be the random suffixes satisfying

$$X_{(i_1-1)} \leq x_1 < X_{(i_1)} \quad \text{and} \quad X_{(i_2-1)} \leq x_2 < X_{(i_2)}.$$

Then, conditional on  $i_1$  and  $i_2$ ,

$$(X_{(1)}, \dots, X_{(i_1-1)}), (X_{(i_1)}, \dots, X_{(i_2-1)}), \text{ and } (X_{(i_2)}, \dots, X_{(n)})$$

become the order statistics in independent samples from three truncated distributions. Since the minimum and the maximum of a sample are asymptotically independent, and  $r_n(x_1)$  depends on  $X_{(i_1-1)}$  and  $X_{(i_1)}$  only, and  $r_n(x_2)$  depends on  $X_{(i_2-1)}$  and  $X_{(i_2)}$  only, we can imitate the earlier part of our proof to show that  $r_n(x_1)$  and  $r_n(x_2)$  are asymptotically independent. ||.

### 3. Remarks on Theorem 2.1.

The important conclusions that can be drawn from Theorem 2.1 are the following:

(i)  $r_n(x)$  is not a consistent estimate of  $r(x)$ ; indeed,  $r_n(x)$  has a limiting non-degenerate distribution.

(ii) While the asymptotic mean of  $r_n(x)$  is  $r(x)$ , we can actually show that

$$E \frac{1}{r_n(x)} \rightarrow 2 \frac{1}{r(x)} .$$

The factor of 2 in the above result is quite surprising, and is reminiscent of the inter-arrival paradox so well explained in Feller [(1966) Vol. 2, p. 11].

(iii) Since  $(r_n(x_1), \dots, r_n(x_k))$  are asymptotically independent, the graph of  $\{r_n(x), x \geq 0\}$  will exhibit wild fluctuations, and will not be a good estimator of the failure rate.

4. A Smoothing of the Naïve Estimate.

The phenomenon described in the previous section is similar to the behavior of the periodogram in estimating the spectrum of a stationary time series. We shall therefore use the standard technique of 'smoothing' the naïve estimate with a 'window' to obtain a consistent and asymptotically normal estimate of the failure rate.

A window is a function  $w(u)$  such that

$$(W1) \quad w(u) = w(-u) \geq 0, \text{ and}$$

$$(W2) \quad \int w(u) du = 1.$$

The window is said to be band-limited if

$$(W3) \quad w(u) = 0, \text{ for } |u| \geq A,$$

and to be bounded, if

$$(W4) \quad 0 \leq w(u) \leq M < \infty$$

The smoothed estimator of the failure rate depends on the window  $w(u)$  and a sequence  $\{b_n\}$  satisfying

$$(B1) \quad b_n \rightarrow 0, \text{ and } nb_n \rightarrow \infty.$$

Without loss of generality, we may assume that

$$(B2) \quad 0 < b_n \leq A.$$

We define the smoothed estimator  $\bar{r}_n(x)$  obtained by smoothing  $r_n$  with the window  $w$  and bandwidth  $2b_n A$  as follows:

$$(4.1) \quad \bar{r}_n(x) = \frac{1}{b_n} \int w\left(\frac{x-s}{b_n}\right) r_n(s) ds$$

$$= \int w(s) r_n(x - b_n s) ds.$$

When  $w$  is a bandlimited window (i.e., satisfies (W3)),  $\bar{r}_n(x)$  will be used as an estimate of  $r(x)$  only for  $x \geq b_n A$ , since the smoothing being done in (4.1) does not include the whole bandwidth of  $w$  when  $x < b_n A$ .

Define

$$R_n(x) = \int_0^x r_n(y) dy = \sum_{j=1}^{i-1} \frac{1}{n-j+1} + \frac{x - X_{(i-1)}}{(n-i+1)(X_{(i)} - X_{(i-1)})}$$

$$\text{for } X_{(i-1)} \leq x < X_{(i)},$$

$$i = 1, \dots, n,$$

$$(4.2) \quad = \sum_{j=1}^{nF_n(x)} \frac{1}{n-j+1} + \theta_n(x) \quad \text{for } x < X_{(n)}$$

where  $0 \leq \theta_n(x) \leq \frac{1}{nF_n(x)}$ .

We may now re-write  $\bar{r}_n(x)$  as

$$\bar{r}_n(x) = \frac{1}{b_n} \int w\left(\frac{x-s}{b_n}\right) dR_n(x).$$

The following lemma shows that  $R_n(x)$  and  $H_n(x) = -\log \bar{F}_n(x)$  (defined only for  $x < X_{(n)}$ ) are uniformly close to each other in bounded intervals.

Lemma 4.1. Fix  $K < \infty$ ; then

$$(4.3) \quad \sup_{0 \leq x \leq K} |R_n(x) - H_n(x)| \leq \frac{3}{2n\bar{F}_n(K)}$$

if  $\bar{F}_n(K) > 0$ ; i.e., if  $K < X_{(n)}$ .

Proof. We shall use the elementary inequality

$$|x + \log(1-x)| \leq \frac{x^2}{2(1-x)} \quad \text{for } 0 \leq x < 1.$$

Note that

$$(4.4) \quad H_n(x) = -\log \bar{F}_n(x) = \sum_{j=1}^{nF_n(x)} \log \frac{n-j+1}{n-j} = - \sum_{j=1}^{nF_n(x)} \log \left( 1 - \frac{1}{n-j+1} \right).$$

From equations (4.2) and (4.4), if  $x \leq K < X_{(n)}$ ,

$$\begin{aligned} |R_n(x) - H_n(x)| &\leq \sum_{j=1}^{nF_n(x)} \left| \frac{1}{n-j+1} + \log \left( 1 - \frac{1}{n-j+1} \right) \right| + \frac{1}{n\bar{F}_n(x)} \\ &= \sum_{j=1}^{nF_n(x)+1} \left| \frac{1}{k} + \log \left( 1 - \frac{1}{k} \right) \right| + \frac{1}{n\bar{F}_n(x)} \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{n-nF_n(K)+1}^n \frac{1}{2} \left(\frac{1}{k}\right)^2 \frac{1}{1 - \frac{1}{n-nF_n(K)+1}} + \frac{1}{n\bar{F}_n(K)} \\
 &\leq \frac{1}{2} \frac{1}{n-nF_n(K)+1} \frac{n-nF_n(K)+1}{n-nF_n(K)} + \frac{1}{n\bar{F}_n(K)} \\
 &= \frac{3}{2} \frac{1}{n\bar{F}_n(K)}. \quad ||.
 \end{aligned}$$

Rice and Rosenblatt (1976) have introduced a class of estimators  $h_n^{(3)}(x)$ , where

$$h_n^{(3)}(x) = \frac{1}{b_n} \int w\left(\frac{x-s}{b_n}\right) dH_n(s).$$

In Lemma 4.2 we will show that our smoothed estimator  $\bar{r}_n(x)$  is uniformly close to  $h_n^{(3)}(x)$  on bounded intervals.

Lemma 4.2. Let  $w$  be a bounded band limited window; that is,  $w$  satisfies (W1), (W2), (W3) and (W4). Let  $w$  also satisfy

$$(W5) \quad \int |w'(x)| dx \leq c < \infty,$$

where  $w'$  is the derivative of  $w$ .

Let the constants  $\{b_n\}$  satisfy (B1) and (B2), and let  $K < \infty$ . Then

$$(4.5) \quad \sup_{0 \leq x \leq K} |\bar{r}_n(x) - h_n^{(3)}(x)| \leq \frac{3c}{2nb_n\bar{F}_n(K+A)}$$

whenever  $K+A \leq X_{(n)}$ .

$$\begin{aligned}
 &\leq \sum_{n-nF_n(K)+1}^n \frac{1}{2} \left(\frac{1}{k}\right)^2 \frac{1}{1 - \frac{1}{n-nF_n(K)+1}} + \frac{1}{n\bar{F}_n(K)} \\
 &\leq \frac{1}{2} \frac{1}{n-nF_n(K)+1} \frac{n-nF_n(K)+1}{n-nF_n(K)} + \frac{1}{n\bar{F}_n(K)} \\
 &= \frac{3}{2} \frac{1}{n\bar{F}_n(K)}. \quad ||.
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where  $w'$  is the derivative of  $w$ .

Let the constants  $\{b_n\}$  satisfy (B1) and (B2), and let  $K < \infty$ . Then

$$(4.5) \quad \sup_{0 \leq x \leq K} |\bar{r}_n(x) - h_n^{(3)}(x)| \leq \frac{3c}{2nb_n \bar{F}_n(K+A)}$$

whenever  $K+A \leq X_{(n)}$ .

Proof: Notice that, for  $x \leq K \leq X_{(n)} - A$

$$\begin{aligned} |\bar{r}_n(x) - h_n^{(3)}(x)| &= \left| \frac{1}{b_n} \int w \frac{x-s}{b_n} d[R_n(x) - H_n(s)] \right| \\ &\leq \left| \frac{1}{b_n} w \left( \frac{x-s}{b_n} \right) [R_n(s) - H_n(s)] \right|_{x-b_n A}^{x+b_n A} \\ &\quad + \frac{1}{b_n} \int_0^{K+A} |R_n(s) - H_n(s)| \left| w' \left( \frac{x-s}{b_n} \right) \right| ds, \end{aligned}$$

(since  $x \leq K$ ,  $b_n \leq A$ , and  $x+b_n \leq K+A$ )

$$\leq \frac{1}{2nb_n \bar{F}_n(K+A)} \text{ from Lemma 4.1 and (W5).} \quad ||$$

Define

$$(4.6) \quad B(x) = \frac{F(x)}{\bar{F}(x)}, \quad B'(x) = \frac{f(x)}{[\bar{F}(x)]^2} = \frac{r(x)}{\bar{F}(x)}$$

and

$$(4.7) \quad r_n^*(x) = \frac{1}{b_n} \int w \left( \frac{x-s}{b_n} \right) dH(s).$$

Let

$$\xi_n(x) = \frac{\sqrt{nb_n}}{\sqrt{B'(x)}} (h_n^{(3)}(x) - r_n^*(x)), \quad b_n A \leq x \leq K,$$

and

$$\bar{\xi}_n(x) = \frac{\sqrt{nb_n}}{\sqrt{B'(x)}} (\bar{r}_n(x) - r_n^*(x)), \quad b_n A \leq x \leq K.$$

In Theorem 4.4, we will show that the process  $\{\xi_n(x), b_n A \leq x \leq K\}$  is uniformly close to a stationary Gaussian process. Because of Lemma 4.2, this same result (Theorem 4.5) will also apply to the process  $\{\bar{\xi}_n(x), b_n A \leq x \leq K\}$ .

In Theorems 4.6, and 4.7, we obtain the asymptotic distribution of  $\max_{b_n A \leq x \leq K} |\xi_n(x)|$  and  $\max_{b_n A \leq x \leq K} |\bar{\xi}_n(x)|$ . In order to make these results useful for obtaining uniform confidence bands for  $r(x)$ ,  $b_n A \leq x \leq K$ , we will have to replace  $r_n^*(x)$  by  $r(x)$  and also replace any unknown quantities that enter in the asymptotic theory by their estimates. This is done in two stages, in Theorem 4.9 and Theorem 4.10. The result in Theorem 4.10 is in the most useful form for applications. Define

$$Z_n(x) = \sqrt{n}(H_n(x) - H(x)).$$

Theorem 4.3. There exists a Gaussian process  $\{Z(x), 0 \leq x \leq K+A\}$  with mean function zero and  $E(Z(x), Z(y)) = B(x)$  for  $x \leq y$ , and such that

$$(4.8) \quad \sup_{0 \leq x \leq K+A} |Z_n(x) - Z(x)| = \frac{D \log n}{\sqrt{n}} \quad \text{w.p. 1,}$$

whenever  $X_{(n)} > K + A$ , where  $D$  is a random variable with  $P(D < \infty) = 1$ .

Proof: In the rest of this paper we use the generic name  $D$  for any random variable with  $P(D < \infty) = 1$ .

From Komlós, Major and Tusnády (1975), there is a Gaussian process  $\{Y(x), 0 \leq x\}$  with mean function zero and  $E(Y(x)Y(y)) = F(x)(1-F(y))$  for  $0 \leq x \leq y$ , and such that

$$\sup_{0 \leq x} |\sqrt{n}(F_n(x) - F(x)) - Y(x)| = \frac{D \log n}{\sqrt{n}} \text{ w.p. } 1.$$

Since  $X_{(n)} > K + A$ , we have for  $0 \leq x \leq K+A$ ,

$$\begin{aligned} Z_n(x) &= -\sqrt{n} \log \frac{\bar{F}_n(x)}{\bar{F}(x)} \\ &= -\sqrt{n} \log \left( 1 + \frac{\bar{F}_n(x) - \bar{F}(x)}{\bar{F}(x)} \right) \\ &= -\sqrt{n} \log \left( 1 - \frac{Y(x)}{\sqrt{n} \bar{F}(x)} + \frac{D \log n}{n \bar{F}(x)} \right) \\ &= \frac{Y(x)}{\bar{F}(x)} + \frac{D \log n}{\sqrt{n}} \\ &= Z(x) + D \frac{\log n}{\sqrt{n}} \end{aligned}$$

where  $Z(x) = \frac{Y(x)}{\bar{F}(x)}$ ,  $0 \leq x \leq K+A$ . It is easy to see that

$\{Z(x), 0 \leq x \leq K+A\}$  is Gaussian with mean 0, and  $E(Z(x)Z(y)) = B(x)$  for  $x \leq y$ . This completes the proof. ||.

Let  $\{W(s), -\infty < s < \infty\}$  denote a Weiner process, and define

$$(4.9) \quad \zeta(\theta) = \int_{-\infty}^{\infty} w(\theta-t)dW(t), \quad 0 \leq \theta < \infty.$$

Then  $\{\zeta(\theta), 0 \leq \theta < \infty\}$  is a stationary Gaussian process with

$$E(\zeta(\theta)) = 0 \quad \text{and} \quad E(\zeta(\theta + \delta)\zeta(\theta)) = \int_{-\infty}^{\infty} w(\delta+t)w(t)dt = \rho(\delta) \quad (\text{say}).$$

We now need to impose further conditions on the distribution function  $F$ :

$$(F1) \quad B''(x) \text{ is bounded on } 0 \leq x \leq K,$$

and

$$(F2) \quad \inf_{0 < x \leq K+A} B'(x) > 0.$$

Theorem 4.4. Let  $w$  satisfy conditions (W1)-(W4). Let (B1), (B2), (F1) and (F2) hold. Then there exists a stationary Gaussian process

$\{\zeta_n(\theta), A \leq \theta \leq K/b_n\}$  which is a restriction of  $\{\zeta(\theta)\}$  for  $A \leq \theta \leq K/b_n$ , such that

$$(4.10) \quad \sup_{b_n A \leq x \leq K} |\xi_n(x) - \zeta_n(x)| = O\left(\frac{\log n}{\sqrt{nb_n} F_n(K+A)} + b_n^{1/2}\right)$$

whenever  $X_{(n)} > K + A$ .

Proof: The proof of this theorem follows by going through steps which are analogous to those used by Bickel and Rosenblatt for their Propositions 2.1 and 2.2. and using the key approximation result in Theorem 4.3. ||.

Theorem 4.5: Theorem 4.4 is true with  $\xi_n(x)$  replaced by  $\bar{\xi}_n(s)$ .

Proof: Immediate from Theorem 4.4 and Lemma 4.2.

||.

In order to obtain the asymptotic distribution of  $\max_{b_n A \leq x \leq K} |\xi_n(x)|$  and  $\max_{b_n A \leq x \leq K} |\bar{\xi}_n(s)|$ , we will let

$$(4.11) \quad \lambda(w) = \int w^2(t) dt,$$

and

$$(4.12) \quad K_1(w) = \frac{w^2(A) + w^2(-A)}{2\lambda(w)}.$$

If  $K_1(w) = 0$ , we shall require that

$$(W6) \quad \int (w'(t))^2 dt < \infty$$

(notice that (W6) implies (W5)), and set

$$(4.13) \quad K_2(w) = \int (w'(t))^2 dt / 2\lambda(w).$$

Let

$$(4.14) \quad c_n = (2 \log (K/b_n))^{1/2}.$$

Define

$$(4.15) \quad \beta_n = c_n / (\lambda/w)^{1/2},$$

and

$$(4.16) \quad \alpha_n = (\lambda(w))^{1/2} [c_n^2 + \log(c_n K_1(w)/\sqrt{2\pi})]/c_n$$

when  $w$  satisfies (W1)-(W5) and  $K_1(w) > 0$ ,

$$(4.17) \quad \alpha_n = (\lambda(w))^{1/2} [c_n^2 + \log(K_2(w)/\pi)]/c_n$$

when  $w$  satisfies (W1)-(W4), (W6) and  $K_1(w) = 0$ .

Theorem 4.6. Let  $M_n = \max_{b_n A \leq x \leq K} |\xi_n(x)|$ . Let (W1)-(W5) hold with  $K_1(w) > 0$

or (W1)-(W4) and (W6) hold with  $K_1(w) = 0$ . Let (B1), (B2), (F1) and (F2) hold. Then

$$(4.18) \quad P\{\beta_n(M_n - \alpha_n) \leq z\} \rightarrow e^{-2e^{-z}}$$

for  $0 < x < \infty$ .

Proof: Follows from Theorem 4.4 and the results on the extrema of a stationary Gaussian process with an autocorrelation function  $\rho(\theta)$  given in Appendix A of Bickel and Rosenblatt. ||

Theorem 4.7. Let  $M_n = \max_{b_n A \leq x \leq K} |\bar{\xi}_n(x)|$ . Theorem 4.6 holds with  $M_n$  replaced by  $\bar{M}_n$ .

Proof: Immediate from Theorems 4.5 and 4.6. ||

In order to obtain results which are useful for applications, we will have to replace  $r_n^*(x)$  in  $\xi_n(x)$  and  $\bar{\xi}_n(x)$  by  $r(x)$ . We shall now indicate the steps which enable us to do this. We impose a further condition

on  $F$  which implies (F2) if  $F(K) > 0$ :

(F3)  $r(x)$  is twice continuously differentiable.

Lemma 4.8. Let (F3) hold. Let (W1), (W2) hold. Then

$$(4.19) \quad \sup_{0 \leq x \leq K} |r_n^*(x) - r(x)| \leq L b_n^2$$

where  $L$  is some finite number.

Proof: Now,

$$\begin{aligned} r_n^*(x) - r(x) &= \frac{1}{b_n} \int w\left(\frac{x-s}{b_n}\right) dH(s) - r(x) \\ &= \int [r(x + t b_n) - r(x)] w(t) dt \\ &= \int_{-A}^A [t b_n r'(x) + \frac{(t b_n)^2}{2} (r''(x) + \gamma(x, t, b_n))] w(t) dt \end{aligned}$$

where  $\gamma(x, t, b_n) \rightarrow 0$  uniformly in  $x, t$

$$= b_n^2 r''(x) \int_{-A}^A t^2 w(t) dt [1 + o(1)],$$

and  $o(1)$  is uniform in  $x$ .

Thus  $\sup_{0 \leq x \leq K} |r_n^*(x) - r(x)| \leq L b_n^2$  where  $0 \leq L < \infty$ .

Let the constants  $b_n$  satisfy the condition

$$(B3) \quad n b_n^5 \log b_n \rightarrow 0.$$

For instance  $b_n = n^{-a}$  with  $\frac{1}{5} < a < \frac{1}{2}$  will satisfy (B1) - (B3).

Theorem 4.9. Under the additional assumption (B3) holds, we may replace  $r_n^*(x)$  by  $r(x)$  in the definitions of  $\xi_n(x)$ ,  $\bar{\xi}_n(x)$ ,  $M_n$  and  $\bar{M}_n$ , and Theorems 4.6 and 4.7 will continue to hold for the new  $M_n$  and  $\bar{M}_n$ .

Proof: Note that

$$\begin{aligned} & \sqrt{2 \log K/b_n} \sqrt{nb_n} (h_n^{(3)}(x) - r(x)) \\ &= \sqrt{2 \log K/b_n} \sqrt{nb_n} (h_n^{(3)}(x) - r_n^*(x)) \\ &+ \sqrt{2 \log K/b_n} \sqrt{nb_n} (r_n^*(x) - r(x)) \\ &= c_n \sqrt{B^*(x)} \xi_n(x) + O(\sqrt{nb_n^5 \log K/b_n}) \\ &= c_n \sqrt{B^*(x)} \xi_n(x) + o(1). \end{aligned}$$

This fact proves the theorem. ||.

Notice that the denominator  $\sqrt{B^*(x)}$  ( $= \sqrt{r(x)/\bar{F}(x)}$ ) in the definitions of  $\xi_n(x)$  and  $\bar{\xi}_n(x)$  is unknown. In Theorem 4.10 we replace  $B^*(x)$  by an estimate to obtain uniform confidence bands for  $r(x)$  using our smoothed estimator  $\bar{r}_n(x)$ . In an analogous manner, we can also obtain uniform confidence bands for  $r(x)$  using  $h_n^{(3)}(x)$ .

Theorem 4.10. Let (W1) - (W5) hold with  $K_1(w) > 0$  or let (W1) - (W4), (W6) hold with  $K_1(w) = 0$ . Let (B1) - (B3), (F1), (F3) hold. Let  $\alpha_n$

and  $\beta_n$  be as defined in (4.15), (4.16) and (4.17). Then

$$(4.20) \quad P \left\{ \bar{r}_n(x) - \left( \frac{\bar{r}_n(x)}{nb_n \bar{F}_n(x)} \right)^{1/2} \left( \frac{z}{\beta_n} + \alpha_n \right) \leq r(x) \right. \\ \left. \leq \bar{r}_n(x) + \left( \frac{\bar{r}_n(x)}{nb_n \bar{F}_n(x)} \right)^{1/2} \left( \frac{z}{\beta_n} + \alpha_n \right) \right\} \rightarrow e^{-2^{-z}},$$

for  $0 < z < \infty$ .

Proof: This theorem follows from the fact that  $\bar{r}_n(x)/\bar{F}_n(x)$  and  $h_n^{(3)}(x)/\bar{F}_n(x)$  are uniformly estimates of  $B'(x)$ .

Remark: To obtain a 100 $\alpha\%$  uniform confidence band for  $r(x)$  for  $b_n A \leq x \leq K$  we use the two functions

$$\bar{r}_n(x) \pm \left( \frac{\bar{r}_n(x)}{nb_n \bar{F}_n(x)} \right)^{1/2} \left( \frac{z}{\beta_n} + \alpha_n \right)$$

with  $z = -\log(-(1/2) \log \alpha)$ .

##### 5. An Illustrative Example With Simulations.

A smoothing window which is quite natural which also turns out to be computationally simple, is the uniform

$$(5.1) \quad w(u) = \frac{1}{2}, \quad |u| < 1.$$

This  $w$  satisfies (W1) - (W5) with  $K_1(w) = \frac{1}{2} > 0$ .

In order to obtain an expression for  $\bar{r}_n(x)$  using this window, we shall make use of the following elementary results.

For any  $x \in [r-1, r]$

$$\frac{1}{r} \leq \int_{r-1}^r \frac{1}{x} dx = \log \frac{r}{r-1} \leq \frac{1}{r-1}.$$

It follows from the above that

$$\sum_{r=n-j+1}^{n-k} \frac{1}{r} \leq \log \frac{n-k}{n-j}$$

and

$$\sum_{r=n-j+2}^{n-k+1} \frac{1}{r-1} \geq \log \frac{n-k+1}{n-j+1}.$$

For any  $x \geq 0$ , and any specified  $b_n$ , let  $Z_1 = (j-1)$  be the number of observations (failures) to the left of  $x - b_n$ ,  $Z_2 = (k-j+1)$  the number of observations between  $x + b_n$  and  $x - b_n$ , and  $Z_3 = (n-k)$  be the number of observations to the right of  $x + b_n$ . Also let

$$\frac{1}{U_i} = \frac{1}{(n-i+1)(X_{(i)} - X_{(i-1)})}, \text{ for } X_{(i-1)} \leq x < X_{(i)},$$

$i = 1, 2, \dots, n$ . If  $W(x) = \int_0^x w(s) ds$ , then

$$\bar{r}_n(x) = \sum_{i=1}^n \int_{X_{(i-1)}}^{X_{(i)}} r_n(x) w(x-s) ds = \sum_{i=1}^n W(x-X_{(i)}) \left( \frac{1}{U_{i+1}} - \frac{1}{U_i} \right).$$

For  $w(\cdot)$  given by equation (5.1), we have

$$\begin{aligned} \bar{r}_n(x) = \frac{1}{2b_n} & \left[ \frac{X_{(j)} - (x-b_n)}{U_j} + \frac{X_{(j+1)} - X_{(j)}}{U_{j+1}} + \dots + \frac{X_{(i)} - X_{(i-1)}}{U_i} \right. \\ & \left. + \dots + \frac{X_{(k)} - X_{(k-1)}}{U_k} + \frac{x+b_n - X_{(k)}}{U_{k+1}} \right] \end{aligned}$$

or that

$$(5.3) \quad \bar{r}_n(x) = \frac{1}{2b_n} \left[ \frac{X_{(j)} - (x-b_n)}{U_j} + \dots + \frac{X_{(i)} - X_{(i-1)}}{(n-i+1)U_{(i)}} + \dots + \frac{x+b_n - X_{(k)}}{U_{k+1}} \right].$$

Based upon the above and the elementary results presented before, we can write

$$\frac{1}{2b_n} \sum_{n-k+1}^{n-j} \frac{1}{r} \leq \bar{r}_n(x) \leq \frac{1}{2b_n} \sum_{n-k}^{n-j+1} \frac{1}{r}.$$

The left hand inequality is obtained by ignoring the first and the last terms of (5.3), whereas the right hand inequality is obtained by setting  $x + b_n = X_{(k+1)}$  and  $x - b_n = X_{(j-1)}$  in the last and the first terms of (5.3), respectively.

Thus

$$\frac{1}{2b_n} \log \frac{n-j+1}{n-k+1} \leq \bar{r}_n(x) \leq \frac{1}{2b_n} \log \frac{n-j+1}{n-k-1}, \text{ or}$$

$$\frac{1}{2b_n} \log \frac{z_2+z_3}{z_3+1} \leq \bar{r}_n(x) \leq \frac{1}{2b_n} \log \frac{z_2+z_3}{z_3-1}.$$

In view of the above, a computationally simple expression for  $\bar{r}_n(x)$  is

$$(5.4) \quad \bar{r}_n(x) \approx \frac{1}{2b_n} \log \frac{z_2+z_3}{z_3}.$$

#### 5.1. Results of a Monte-Carlo Experiment.

In this section we shall summarize our experience with the estimator given by equation (5.4), based on a Monté Carlo experiment.

Random samples of size  $n$  ( $= 10, 15, 20$  and  $50$ ) were generated from an exponential distribution with scale parameter  $1.0$ , and a Weibull distribution with shape parameter  $2.0$ , respectively. For each sample size we obtain  $\bar{r}_n(x)$  using a uniform window with  $b_n = \frac{1}{n^a}$ , for  $a = 0.05, 0.10, 0.15, 0.20, 0.25,$  and  $0.75$ . We repeat this procedure  $500$  times for each of the specified  $24$  combinations of sample size and window width. Based upon the results of these  $500$  trials, we calculate at each point  $x$ , the average, the standard error, and the root mean square error of  $\bar{r}_n(x)$ . These statistics

are summarized in Table 5.1 for the exponential distribution at the point  $x = 1.0$ , and in Table 5.2 for the Weibull distribution at the point  $x = 0.5$ .

In Figure 5.1 we show a plot of the average value of  $\bar{r}_n(x)$  at each point  $x$ , based on  $n = 10$  and  $a = 0.75$  for the exponential distribution, whereas in Figure 5.2 we show a similar plot for  $n = 50$ . In Figures 5.3 and 5.4 we give plots analogous to those in Figures 5.1 and 5.2, respectively, but for the Weibull distribution.

On the basis of the above described experiment, we can state the following by way of conclusion:

i) For samples as small as 10 or 15, the smoothed estimator obtained by using windows of small width is generally unbiased; the bias increases with the width of the window. This phenomenon is especially true for points in the middle of the range of  $x$ .

ii) The standard error decreases with the width of the window.

iii) It appears that towards the end points of the range of  $x$ , a smoothed estimator based on a narrower window width performs better than one based on a wider window width.

The above comments suggest the possibility of using windows of varying widths. A version of this strategy has been considered by Miller and Singpurwalla (1978). For further details on the Monté-Carlo experiment, we refer the reader to Chandra (1977).

TABLE 5.1

SAMPLING PROPERTIES OF SMOOTHED ESTIMATOR OF THE FAILURE RATE  
AT THE POINT  $x = 1.0$  FROM AN EXPONENTIAL DISTRIBUTION

SAMPLE SIZE	WINDOW WIDTH	MEAN	STANDARD ERROR	ROOT MEAN SQUARED ERROR
10	0.356	1.0649	1.0259	1.0280
	1.125	0.9362	0.5126	0.5166
	1.262	0.9178	0.4854	0.4923
	1.416	0.8518	0.4517	0.4754
	1.589	0.8240	0.4608	0.4932
	1.782	0.7710	0.4396	0.4961
15	0.262	1.1780	1.1004	1.1147
	1.016	1.0373	0.5153	0.5167
	1.164	1.0248	0.4376	0.4383
	1.332	1.0452	0.4327	0.4350
	1.526	0.9704	0.3974	0.3985
	1.747	0.9306	0.4019	0.4079
20	0.212	1.0978	0.8748	0.8802
	0.946	1.0537	0.4501	0.4533
	1.099	1.0751	0.4123	0.4191
	1.276	1.0353	0.3658	0.3675
	1.582	1.0376	0.3606	0.3626
	1.722	0.9966	0.3452	0.3452
50	0.106	1.0145	0.7258	0.7259
	0.752	1.0237	0.2835	0.2845
	0.915	1.0183	0.2599	0.2605
	1.112	1.0479	0.2449	0.2495
	1.353	1.0280	0.2284	0.2301
	1.645	1.0251	0.1982	0.1998

TABLE 5.2

SAMPLING PROPERTIES OF SMOOTHED ESTIMATOR OF THE FAILURE RATE AT THE POINT  $x = 0.5$  FROM A WEIBULL DISTRIBUTION

SAMPLE SIZE	WINDOW WIDTH	MEAN	STANDARD ERROR	ROOT MEAN SQUARED ERROR
10	0.356	1.0459	0.6402	0.6419
	1.125	1.0769	0.4614	0.4678
	1.262	1.0314	0.4598	0.4609
	1.416	0.9773	0.4189	0.4195
	1.589	0.9309	0.4644	0.4695
	1.782	0.8130	0.4744	0.5099
15	0.262	1.0530	0.5870	0.5894
	1.016	1.0746	0.4022	0.4091
	1.164	1.0649	0.3874	0.3928
	1.332	1.0995	0.3783	0.3912
	1.526	1.0549	0.3818	0.3857
	1.747	1.0063	0.4144	0.4144
20	0.212	0.9851	0.5809	0.5811
	0.946	1.0422	0.3224	0.3251
	1.099	1.0633	0.3245	0.3307
	1.276	1.0494	0.2945	0.2987
	1.582	1.0855	0.3350	0.3458
	1.722	1.0816	0.3479	0.3574
50	0.106	0.9893	0.4915	0.4917
	0.752	1.0270	0.2043	0.2061
	0.915	1.0230	0.1997	0.2010
	1.112	1.0154	0.1912	0.1918
	1.353	1.0339	0.1958	0.1988
	1.645	0.0867	0.1876	0.2067

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Figure 5.1. Mean value of the smoothed estimator of the failure rate for samples of size 10 from an exponential distribution (Window width 0.356)

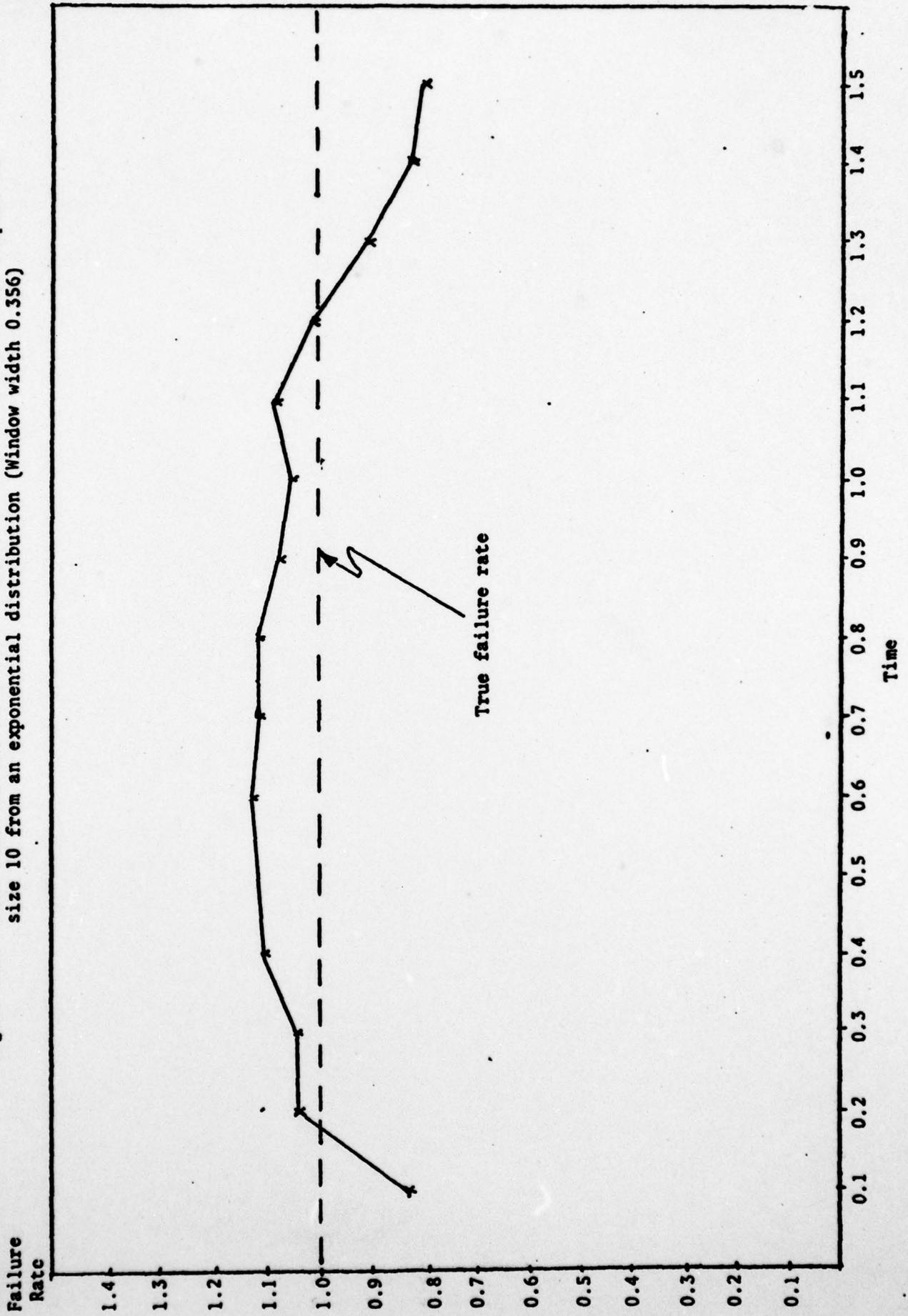


Figure 5.2. Mean value of the smoothed estimator of the failure rate for sample of size 50 from an exponential distribution (Window width 0.106)

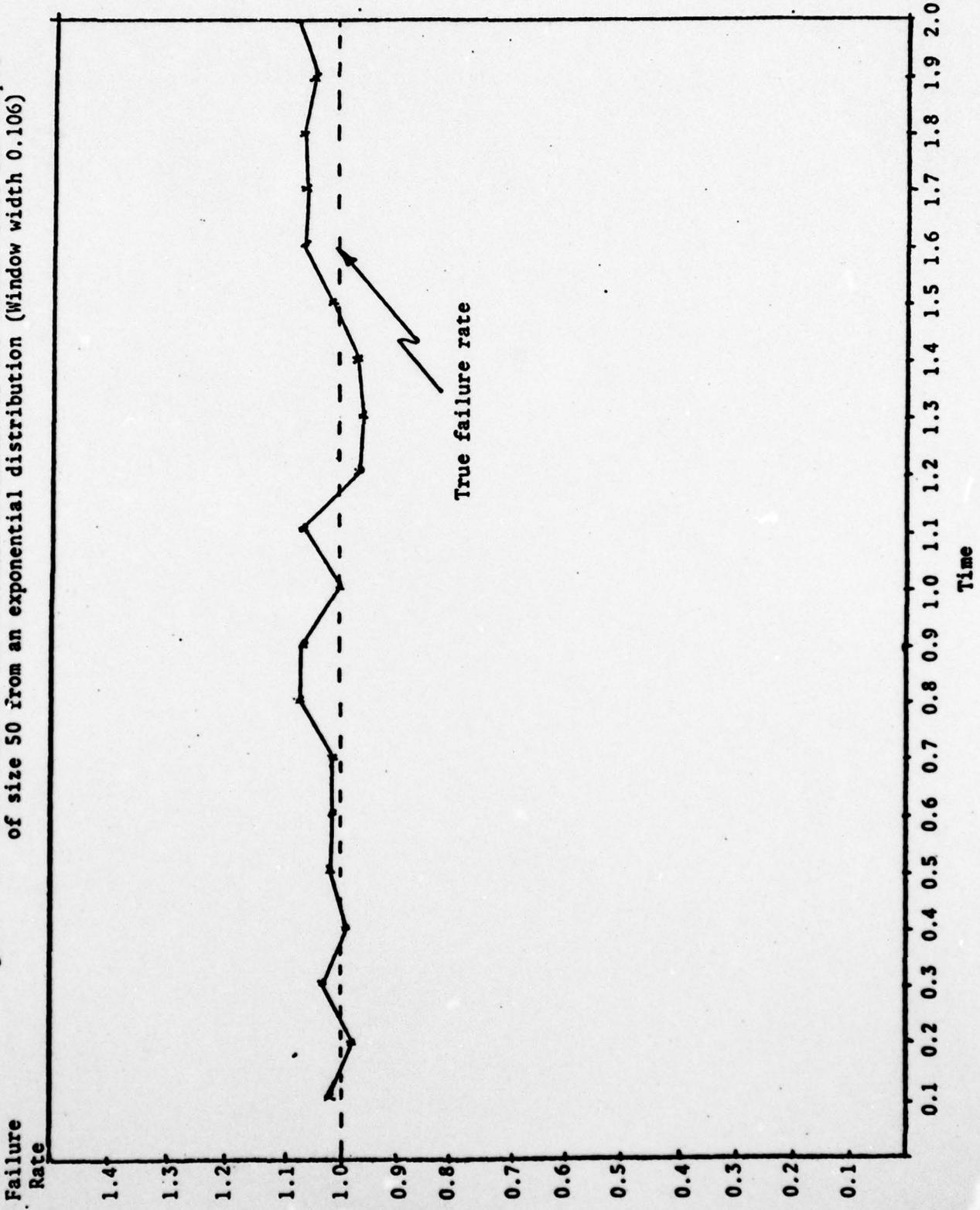


Figure 5.3. Mean value of the smoothed estimator of the failure rate for samples of size 10 from a Weibull distribution (Window width 0.356)

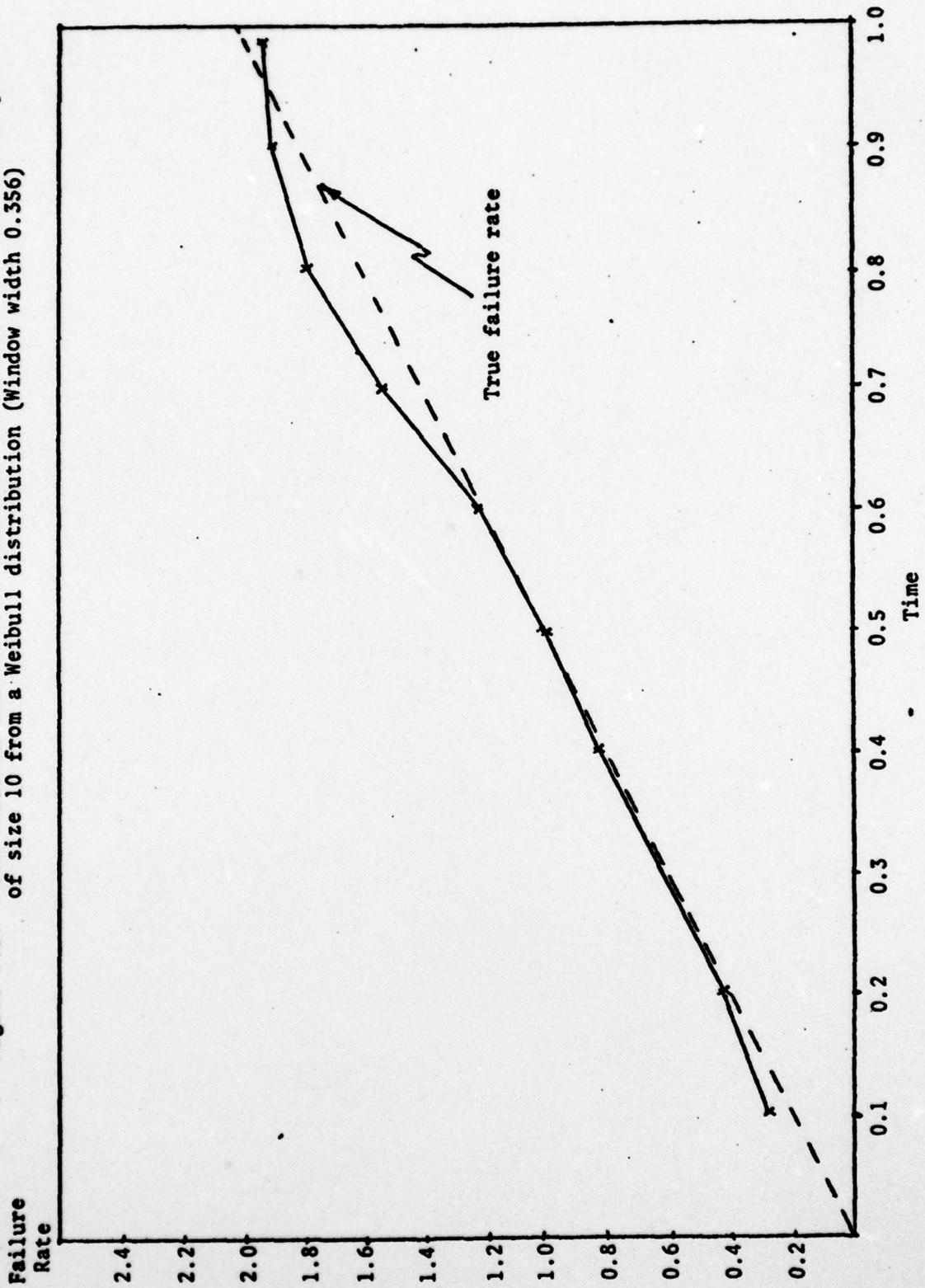


Figure 5.4. Mean value of the smoothed estimator of the failure rate for samples of size 50 from a Weibull distribution (Window width 0.106)

